Fourier Analysis

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Review.

Def. A function $f : \mathbb{R} \to \mathbb{C}$ is said to be of moderate decrease if

① f is cts on $\mathbb{R}$.

② $\exists\ A > 0$ such that

$$|f(x)| \leq \frac{A}{1+x^2} \quad \text{for all } x \in \mathbb{R}.$$

Write

$$\mathcal{M}(\mathbb{R}) = \{ f : f \text{ is of moderate decrease} \}.$$

Def. (improper integration)

Let $f \in \mathcal{M}(\mathbb{R})$. We define

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{N\to\infty} \int_{-N}^{N} f(x)\,dx.$$

$$= \lim_{N,M\to+\infty} \int_{-N}^{M} f(x)\,dx$$

Lemma. We write

$$L(f) = \int_{-\infty}^{\infty} f(x)\,dx \quad \text{for } f \in \mathcal{M}(\mathbb{R}).$$

Then

① L is linear, i.e.

$$L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$$

for $f, g \in \mathcal{M}(\mathbb{R})$ and $\alpha, \beta \in \mathbb{C}$.

② L is translation invariant.

$$\int_{-\infty}^{\infty} f(x+h)\,dx = \int_{-\infty}^{\infty} f(x)\,dx, \quad \forall h \in \mathbb{R}.$$

③ Scaling under dilation: $\forall \delta > 0$,

$$\delta \int_{-\infty}^{\infty} f(\delta x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

④ Absolute continuity.

$$\lim_{h \to 0} \int_{-\infty}^{\infty} |f(x+h) - f(x)|\,dx = 0.$$

§ 5.2 Fourier transform on $\mathbb{R}$.

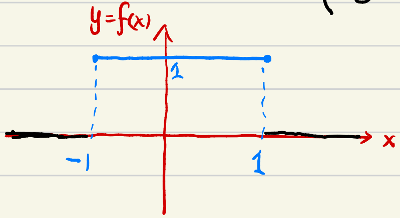
Def. Let $f \in M(\mathbb{R})$. The Fourier transform of f is

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) \cdot e^{-2\pi i \xi x} \, dx, \quad \xi \in \mathbb{R}.$$

Here: $\hat{f} : \mathbb{R} \to \mathbb{C}$.

- $f(x) e^{-2\pi i \xi x} \in M(\mathbb{R})$ for every $\xi \in \mathbb{R}$.
- $$|\hat{f}(\xi)| = \left| \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx \right| \leq \int_{-\infty}^{\infty} |f(x)| \, dx < \infty.$$

Example 1. Let $f(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$



For $\xi \neq 0$,

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} dx$$

$$= \int_{-1}^{1} 1 \cdot e^{-2\pi i \xi x} dx$$

$$= \left. \frac{e^{-2\pi i \xi x}}{-2\pi i \xi} \right|_{x=-1}^{1}$$

$$= \frac{e^{-2\pi i \xi} - e^{2\pi i \xi}}{-2i \ (\pi \xi)}$$

$$= \frac{\sin(2\pi\xi)}{\pi\xi}.$$

If $\xi = 0$, we have

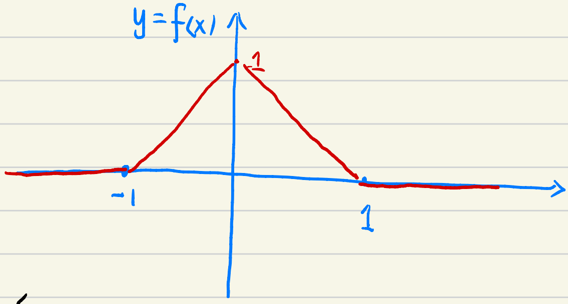
$$\hat{f}(0) = \int_{-1}^{1} 1 \, dx = 2$$

So

$$\hat{f}(\xi) = \begin{cases} \dfrac{\sin(2\pi\xi)}{\pi\xi} & \text{if } \xi \neq 0 \\ 2 & \text{if } \xi = 0. \end{cases}$$

Example 2. Let

$$f(x) = \begin{cases} 1-|x| & \text{if} \quad |x| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$



For $\xi \neq 0$,

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \xi x}\, dx$$

$$= \int_{-1}^{1} (1-|x|)\, e^{-2\pi i \xi x}\, dx$$

$$= \int_{0}^{1} (1-x)\, e^{-2\pi i \xi x}\, dx + \int_{-1}^{0} (1+x)\, e^{-2\pi i \xi x}\, dx$$

$$= (1-x)\frac{e^{-2\pi i \xi x}}{-2\pi i \xi}\Bigg|_0^1 - \int_0^1 \frac{e^{-2\pi i \xi x}}{-2\pi i \xi} \cdot \underline{(-1)}\, dx$$

$$(1+x)\frac{e^{-2\pi i \xi x}}{-2\pi i \xi}\Bigg|_{-1}^{0} - \int_{-1}^{0} \frac{e^{-2\pi i \xi x}}{-2\pi i \xi}\cdot 1 \, dx$$

$$= \cancel{\frac{1}{2\pi i \xi}} + \frac{e^{-2\pi i \xi x}}{(-2\pi i \xi)^2}\Bigg|_{0}^{1}$$

$$+ \cancel{\frac{1}{-2\pi i \xi}} - \frac{e^{-2\pi i \xi x}}{(-2\pi i \xi)^2}\Bigg|_{-1}^{0}$$

$$= \frac{e^{-2\pi i \xi} - 1}{-4\pi^2 \xi^2} - \frac{1 - e^{2\pi i \xi}}{-4\pi^2 \xi^2}$$

$$= \frac{2 - e^{-2\pi i \xi} - e^{2\pi i \xi}}{4\pi^2 \xi^2}$$

$$= \frac{2 - 2\cos(2\pi \xi)}{4\pi^2 \xi^2} = \frac{\sin^2(\pi \xi)}{\pi^2 \xi^2}$$

when $\xi = 0$,

$$\hat{f}(0) = \int_{-1}^{1} (1 - |x|)\, dx = 1.$$

§ 5.3. Some basic properties of Fourier transform.

Prop 1. Let $f \in \mathcal{M}(\mathbb{R})$. Then the following hold:

(1) $$f(x+h) \xrightarrow{\mathcal{F}} \hat{f}(\xi) \cdot e^{2\pi i \xi h} \qquad \forall\, h \in \mathbb{R}.$$

(2) $$f(x) \cdot e^{-2\pi i h x} \xrightarrow{\mathcal{F}} \hat{f}(\xi + h), \qquad \forall\, h \in \mathbb{R}.$$

(3) Let $\delta > 0$. Then

$$f(\delta x) \xrightarrow{\mathcal{F}} \frac{\hat{f}\left(\frac{\xi}{\delta}\right)}{\delta}.$$

(4) Suppose $f' \in \mathcal{M}(\mathbb{R})$. Then

$$f'(x) \xrightarrow{\mathcal{F}} \hat{f}(\xi) \cdot (2\pi i \xi)$$

(5) Suppose $x f(x) \in \mathcal{M}(\mathbb{R})$. Then

$$f(x) \cdot (-2\pi i x) \xrightarrow{\mathcal{F}} \frac{d \hat{f}(\xi)}{d \xi}$$